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Conservative multi-exponential scheme for solving the direct Zakharov–Shabat scattering problem

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The direct Zakharov–Shabat scattering problem has recently gained significant attention in various applications of fiber optics. The development of accurate and fast algorithms with low computational complexity to solve the Zakharov–Shabat problem (ZSP) remains an urgent problem in optics. In this Letter, a fourth-order multi-exponential scheme is proposed for the Zakharov–Shabat system. The construction of the scheme is based on a fourth-order three-exponential scheme and Suzuki factorization. This allows one to apply the fast algorithms with low complexity to calculate the ZSP for a large number of spectral parameters. The scheme conserves the quadratic invariant for real spectral parameters, which is important for various telecommunication problems related to information coding.

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Introduction. In 1971, Zakharov and Shabat showed that the nonlinear Schrödinger equation (NLSE),

$$i\frac{\partial q}{\partial z}+\frac{\sigma}{2}\frac{\partial^2 q}{\partial t^2}+|q|^2 q=0, \quad \sigma=\pm 1, \tag{1}$$

can be integrated by the inverse problem method [or so-called nonlinear Fourier transform (NFT)] previously applied to the Korteweg de Vries equation [1]. After that, interest in the NLSE arose in all areas of physics connected with wave systems, because the NLSE describes the envelope for narrow wave beams. In 1973, Hasegawa and Tappert numerically investigated the NLSE with respect to the propagation of light pulses in optical fibers [2]. They proposed using solitons as an information carrier for fiber lines with anomalous dispersion at $\sigma=1$. For normal dispersion at $\sigma=-1$, solitons do not exist, as is well known. The NLSE has found widespread use in telecommunication applications. In the past few years, some new NFT-based approaches have been actively investigated to compensate for fiber nonlinearity and to overcome the limitations of linear transmission methods imposed by nonlinearity [3–6].

On the other hand, attempts to create fast numerical algorithms for solving the inverse scattering problem for the NLSE have not stopped. Such methods are combined under the general name fast NFT (FNFT) [7–11] and offer a new approach for numerical scheme construction having a low computational complexity $\mathcal{O}(M\ \log^2 M)$, where M is the number of signal samples. Similar to the fast Fourier transform (FFT), these algorithms can increase computational speed in comparison with traditional approaches with a complexity $\mathcal{O}(M^2)$.

In this Letter, we propose a special fourth-order numerical method for solving the direct Zakharov–Shabat problem (ZSP) and a fast algorithm for its numerical implementation. The main advantage of the presented scheme is the conservation of the quadratic invariant for real spectral parameters, even in the fast version. This is the first proposed fast scheme with such a property, to the best of our knowledge. Moreover, the accuracy of the proposed scheme does not degrade when switching to the fast variant for continuous spectrum computation, unlike most other FNFT algorithms. Also, the fast variant does not introduce a large error for big spectral parameter values for either sign of dispersion. The quadratic invariant conservation by a numerical scheme allows calculating precisely the reflection coefficient, which is valuable for various telecommunication problems connected with NFT-based coding schemes (e.g., nonlinear frequency-division multiplexing [3] and b-modulation [4]) and long-haul transmissions. Also, the proposed scheme works for uniform grids, which, together with the quadratic invariant conservation, makes it attractive for telecommunication problems.

Direct spectral ZSP with the complex spectral parameter ζ for the NLSE Eq. (1) can be written as an evolutionary system:

$$\frac{d\Psi(t)}{dt}=Q(t)\Psi(t), \tag{2}$$

where $q=q(t,z_0)$ is the initial field for the NLSE at the point z_0, which is the potential in the ZSP, and

$$\Psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \quad Q(t) = \begin{pmatrix} -i\zeta & q \\ -\sigma q^* & i\zeta \end{pmatrix}.$$

Here z_0 plays the role of the parameter, and we will omit it. For details, we refer to the numerous literatures, in particular, [12].

Moreover, the system Eq. (2) can be written in the gradient form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = KD \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = K \begin{pmatrix} \frac{\partial H}{\partial \psi_1^*} \\ \frac{\partial H}{\partial \psi_2^*} \end{pmatrix}, \tag{3}$$

where $H = |\psi_1|^2 + \sigma|\psi_2|^2$,

$$K = \begin{pmatrix} -i\zeta & \sigma q \\ -\sigma q^* & i\sigma\zeta \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}. \tag{4}$$

For real $\zeta = \xi$, the matrix K before the gradient becomes anti-Hermitian $K = -K^\dagger$ for any $\sigma = \pm 1$, and therefore, the system Eq. (2) will conserve the quadratic form H.

Assuming that $q(t)$ decays rapidly when $t \to \pm\infty$, the Jost functions for ZSP Eq. (2) can be derived as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\zeta t} \\ 0 \end{pmatrix} [1 + o(1)], \quad t \to -\infty. \tag{5}$$

Then we obtain the scattering coefficients $a(\xi)$ and $b(\xi)$ as:

$$a(\xi) = \lim_{t\to\infty} \psi_1(t, \xi)e^{i\xi t}, \quad b(\xi) = \lim_{t\to\infty} \psi_2(t, \xi)e^{-i\xi t}. \tag{6}$$

The continuous spectrum is determined by the reflection coefficient $r(\xi) = b(\xi)/a(\xi)$, $\xi \in \mathbb{R}$. For anomalous dispersion, the discrete spectrum exists. It is defined by eigenvalues of ZSP that are zeros of $a(\zeta) = 0$, where ζ is a complex number with a positive imaginary part. The discrete spectrum consists of the eigenvalues ζ_k and associated phase coefficients $r_k = b(\zeta)/a'(\zeta)|_{\zeta=\zeta_k}$, where $a'(\zeta) = \frac{da(\zeta)}{d\zeta}$.

For real values of the spectral parameter $\zeta = \xi$, we have the quadratic invariant H. Taking into account the conditions Eq. (5), we get the same condition $H = 1$ for $\sigma = \pm 1$.

Summing up, we solve a linear system of the form Eq. (2) with the matrix $Q(t)$ linearly dependent on the complex function $q(t)$. The numerical implementation of the continuous function $q(t)$ is a discrete function $q_n = q(t_n)$, which is defined at the integer nodes t_n of the uniform grid with the step τ. Since we are considering a finite time interval, we will solve the problem on the interval $[-L, L]$ with the total number of points equal to $M + 1$; the grid step in this case is $\tau = 2L/M$, and $t_n = -L + \tau n$, where $n = 0, \ldots, M$.

The main features of the computational problem can be found in Ref. [12]. Briefly, the unknown function Ψ must be calculated on a uniform grid; Dahlquist's second barrier restricts the application of multistep methods [13,14]; matrix exponentials can be easily calculated for matrices 2×2; it is necessary to solve the Zakharov–Shabat system for a large number of spectral parameter values ζ at a fixed potential $q(t)$.

Scheme. Previously, we found the necessary conditions for the existence of a one-step fourth-order scheme $\Psi(t_n + \tau/2) = T_n \Psi(t_n - \tau/2)$ in the form of the Taylor series for the transition matrix T_n. We obtained several fourth-order schemes that exactly conserve the quadratic invariant H. However, only one scheme in the form of three exponentials is convenient for fast computation [12]:

$$T_n = e^{\left\{\frac{\tau^2}{12} Q_n^{(1)} + \frac{\tau^3}{48} Q_n^{(2)}\right\}} e^{\tau Q_n} e^{\left\{-\frac{\tau^2}{12} Q_n^{(1)} + \frac{\tau^3}{48} Q_n^{(2)}\right\}}, \tag{7}$$

where $Q_n = Q(t_n)$ and $Q_n^{(k)}$, $k = 1, 2$, are the k-th derivatives of the matrix Q approximated by central finite differences of the second order.

The FNFT algorithm is based on the representation of the transition matrix as a polynomial of a matrix exponential, depending on the spectral parameter. Then the fast algorithms are applied to calculate the product of polynomials. The scheme (7) contains the spectral parameter only in the central exponential. To construct the fast algorithm, we must split the matrix

$$Q = A + B, \ A = \begin{pmatrix} -i\zeta & 0 \\ 0 & i\zeta \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & q \\ -\sigma q^* & 0 \end{pmatrix} \tag{8}$$

and express $\exp(\tau Q)$ in the form of a polynomial in exponentials of A and B with rational weights. For example, one can use the expansions for $\exp(\tau Q)$ suggested in Ref. [15]. However, representing the sum of the product of exponentials does not guarantee the exact conservation of the invariant H. In order for the scheme to be suitable for the fast algorithm and conserve the invariant H, it suffices to represent the matrix $\exp(\tau Q)$ as

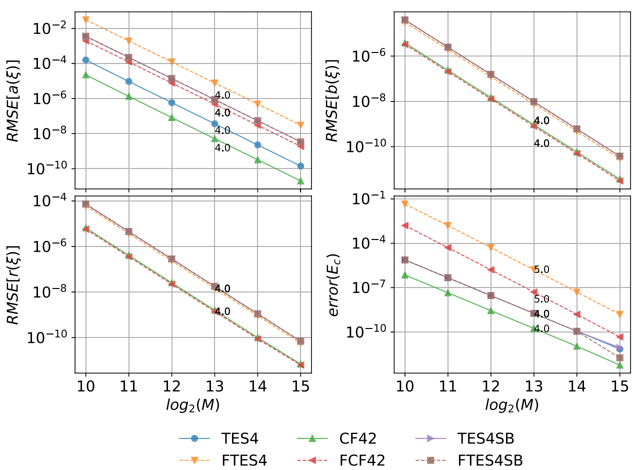

Fig. 1. Continuous spectrum errors for the chirped hyperbolic secant in the case of anomalous dispersion $\sigma = 1$.

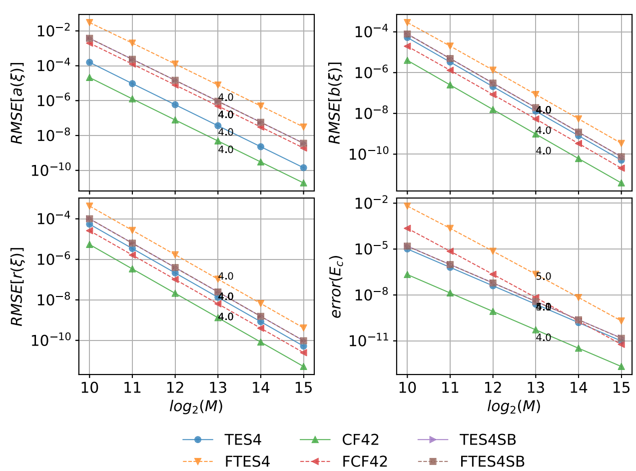

Fig. 2. Continuous spectrum errors for the chirped hyperbolic secant in the case of normal dispersion $\sigma = -1$.

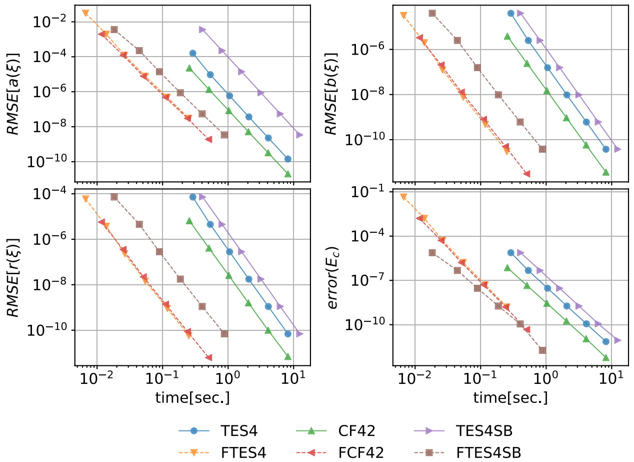

Fig. 3. Continuous spectrum errors depending on the execution time trade-off for the chirped hyperbolic secant and anomalous dispersion $\sigma = 1$.

the product of exponentials of A and B with real rational coefficients. Since for $\sigma = 1$ the matrices A and B are Hermitian, then in this case, each exponent will be unitary, and the resulting scheme will conserve the quadratic invariant H. Conserving H also takes place for $\sigma = -1$. Details can be found in Ref. [12]. The rationality condition for weight coefficients provides an opportunity to represent the transition matrix in the form of the ratio of two polynomials in $\exp(A)$.

Suzuki factorization. Since the scheme (7) has a fourth order of accuracy in τ, it is necessary to have factorization of the same order. In addition, factorization should be suitable for a fast algorithm, i.e., have rational weights. An example of such factorization is given in Ref. [16]:

$$e^{\tau(A+B)} = e^{\frac{7}{48}\tau B} e^{\frac{1}{3}\tau A} e^{\frac{3}{8}\tau B} e^{-\frac{1}{3}\tau A} e^{-\frac{1}{48}\tau B}$$

$$\times\, e^{\tau A} e^{-\frac{1}{48}\tau B} e^{-\frac{1}{3}\tau A} e^{\frac{3}{8}\tau B} e^{\frac{1}{3}\tau A} e^{\frac{7}{48}\tau B}. \qquad (9)$$

We introduce the notation $Z = \exp(-\frac{i}{3}\tau\zeta)$, then the three exponents participating in this expansion take the forms

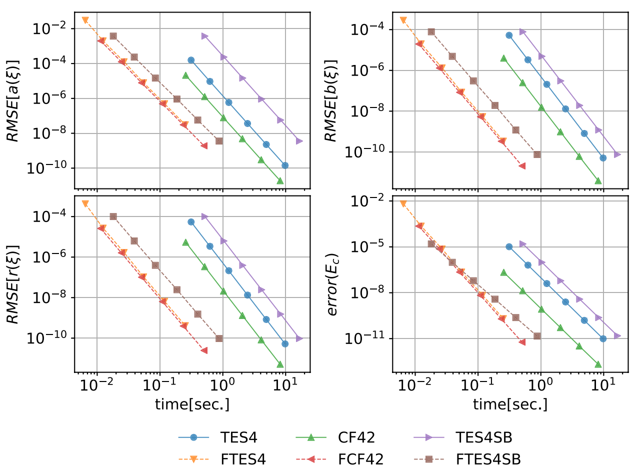

Fig. 4. Continuous spectrum errors depending on the execution time trade-off for the chirped hyperbolic secant and normal dispersion $\sigma = -1$.

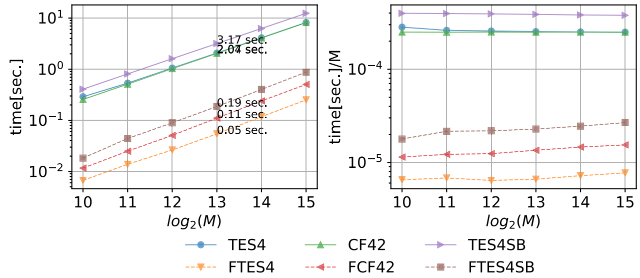

Fig. 5. Execution times for different algorithms in the case of anomalous dispersion $\sigma = 1$.

$$e^{\frac{1}{3}\tau A} = \begin{pmatrix} Z & 0 \\ 0 & Z^{-1} \end{pmatrix} = Z^{-1}\begin{pmatrix} Z^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (10)$$

$$e^{-\frac{1}{3}\tau A} = \begin{pmatrix} Z^{-1} & 0 \\ 0 & Z \end{pmatrix} = Z^{-1}\begin{pmatrix} 1 & 0 \\ 0 & Z^2 \end{pmatrix}, \qquad (11)$$

$$e^{\tau A} = \begin{pmatrix} Z^3 & 0 \\ 0 & Z^{-3} \end{pmatrix} = Z^{-3}\begin{pmatrix} Z^6 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (12)$$

Thus, the right-hand side of Eq. (9) is a rational function $\frac{S(Z)}{Z^7}$, where $S(Z)$ is a polynomial not higher than 14 deg in Z.

Since Z is included only in the square, it is possible to introduce the variable $W = Z^2$, then the rational function takes the form $\hat{S}(W)/W^{\frac{7}{2}}$, where $\hat{S}(W)$ is a polynomial of deg 7 or less in W. The denominator is taken out and calculated independently. It should be noted that besides factorization Eq. (9), the symmetrical representation can be applied when the matrices A and B are interchanged. Such factorization leads to the polynomial $\hat{S}(W)$ of deg 52, which is more computationally difficult and is less accurate, so it is not considered here.

Thus, we represent a new scheme with Suzuki factorization that is based on the transition matrix (7), where the central exponential is factorized by Eq. (9).

Numerical examples. The presented scheme implementation was based on the FNFT software library [17]. It was compared with the triple-exponential scheme without factorized exponential Eq. (7) (TES4) [12] and the fourth-order commutator-free quasi-Magnus exponential scheme $\mathrm{CF}_2^{[4]}$ with FFT interpolation of the signal [11]. These algorithms conserve the quadratic invariant H for real spectral parameters ξ. We have considered both variants of the scheme with Suzuki factorization: conventional (TES4SB) and fast (FTES4SB). The last letter in the scheme name denotes the decomposition type: TES4SA denotes the scheme with the exponential with matrix

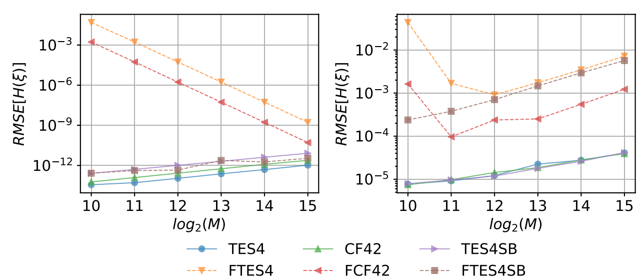

Fig. 6. Maximum value of the quadratic invariant H conservation error for (a) anomalous dispersion $\sigma = 1$ and (b) normal dispersion $\sigma = -1$.

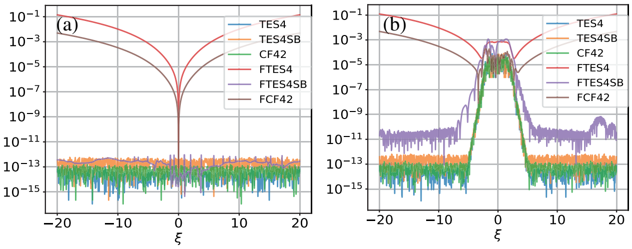

Fig. 7. Quadratic invariant conservation error $|1-|\psi_1|^2-\sigma|\psi_2|^2|$ depending on the spectral parameter ξ: (a) $\sigma=1$, (b) $\sigma=-1$.

A at the edges of Suzuki decomposition, while TES4SB refers to scheme (9). Moreover, we compared the proposed scheme with the fast variant of $CF_2^{[4]}$ scheme ($FCF_2^{[4]}$) and the fast variant of TES4 scheme (FTES4) constructed by fourth-order decomposition of the exponential (Eq. (20) in Ref. [15]), which provides the best computational speed but does not conserve the quadratic invariant. Scheme $FCF_2^{[4]}$ was considered in Ref. [11], while FTES4 was not presented elsewhere before.

A model signal was considered in the form of a chirped hyperbolic secant $q(t)=A[\text{sech}(t)]^{1+iC}$ with the following parameters: $A=5.2$, $C=4$ for both anomalous and normal dispersion. The detailed analytical expressions of the spectral data for these types of potentials can be found in Ref. [12].

We present the numerical errors of calculating the spectral data for a continuous spectrum only, because of focusing on the conservation of the invariant H for real spectral parameters ξ. To find the calculation errors of the continuous spectrum energy E_c, the following formula was used:

$$\text{err}[E_c]=\frac{|E_c^{\text{comp}}-E_c^{\text{exact}}|}{\phi_0},\quad \phi_0=\begin{cases}E_c^{\text{exact}}, & \text{if } |E_c^{\text{exact}}|>1\\ 1, & \text{otherwise.}\end{cases}$$

For the continuous spectrum, we calculated the root mean squared error:

$$\text{RMSE}[\phi]=\sqrt{\frac{1}{N}\sum_{j=1}^{N}\frac{|\phi^{\text{comp}}(\xi_j)-\phi^{\text{exact}}(\xi_j)|^2}{|\phi_0(\xi_j)|^2}},$$

$$\phi_0=\begin{cases}\phi^{\text{exact}}(\xi_j), & \text{if} |\phi^{\text{exact}}(\xi_j)|>1\\ 1, & \text{otherwise,}\end{cases} \tag{13}$$

where ϕ can represent $a(\xi)$, $b(\xi)$, $r(\xi)$ or $|H^{\text{comp}}(\xi)-H^{\text{exact}}(\xi)|$. Here we assume the spectral parameter $\xi\in[-20,20]$ with the total number of points $N=1025$ to compare the schemes by runtime depending only on the number of integration steps.

Figures 1 and 2 present the errors calculated using the schemes under consideration. The scheme $CF_2^{[4]}$ in this particular case demonstrates an error for the coefficient $b(\xi)$ and the reflection coefficient $r(\xi)$ an order of magnitude smaller than other algorithms, in both conventional and fast variants. But it should be noted that due to the quadratic invariant conservation, the accuracy of the proposed scheme TES4SB does not degrade when switching to the fast variant (FTES4SB), unlike other algorithms. Also, all schemes demonstrate the fourth order for the model signal and both signs of dispersion, except schemes FTES4 and $FCF_2^{[4]}$, which provide a fifth-order error decrease for the energy value E_t. The order is marked in Figs. 1 and 2 as 4.0 or 5.0.

The efficiency of the schemes is compared in Figs. 3 and 4. The fast variant of the proposed algorithm FTES4SB demonstrates the worst speed when getting the desired error value across all considered fast schemes for both signs of dispersion. It can be explained by the degree of polynomial used for the transition matrix representation (7 for FTES4SB, 4 for $FCF_2^{[4]}$, and 2 for FTES4). Of course, due to an asymptotic complexity of fast methods [7], one can determine the temporal grid size M for a fixed number of spectral parameter values N when the speed and efficiency of the fast schemes become comparable with conventional algorithms, which is demonstrated in Fig. 5.

The conservation properties of the schemes are considered in Figs. 6 and 7. Schemes FTES4 and $FCF_2^{[4]}$ introduce a large error for big spectral parameter values $|\xi|$ for both signs of dispersion. All other algorithms including the proposed scheme in a fast variant (FTES4SB) demonstrate good conservation of the quadratic invariant H for anomalous dispersion. In the case of normal dispersion, an error sufficiently increases due to the subtraction of large modulo quantities. All conventional schemes are comparable in the magnitude of the error. The accuracy of the proposed scheme reaches the close value of the error, though the fast computation technique caused an increase in error up to two orders of magnitude for normal dispersion.

In conclusion, we have developed a new multi-exponential scheme based on our three-exponential scheme and Suzuki decomposition, which allows fast computation and conserves the quadratic invariant for the real spectral parameter. The scheme consists of 13 matrix exponentials and has the fourth order of approximation. Also, it works for uniform grids, which, together with the quadratic invariant conservation, makes the proposed scheme attractive for telecommunication problems.

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