New Approaches to Coding Information using Inverse Scattering Transform

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Remarkable mathematical properties of the integrable nonlinear Schrödinger equation (NLSE) can offer advanced solutions for the mitigation of nonlinear signal distortions in optical fiber links. Fundamental optical soliton, continuous, and discrete eigenvalues of the nonlinear spectrum have already been considered for the transmission of information in fiber-optic channels. Here, we propose to apply signal modulation to the kernel of the Gelfand-Levitan-Marchenko equations that offers the advantage of a relatively simple decoder design. First, we describe an approach based on exploiting the general N-soliton solution of the NLSE for simultaneous coding of N symbols involving $4 \times N$ coding parameters. As a specific elegant subclass of the general schemes, we introduce a soliton orthogonal frequency division multiplexing (SOFDM) method. This method is based on the choice of identical imaginary parts of the N-soliton solution eigenvalues, corresponding to equidistant soliton frequencies, making it similar to the conventional OFDM scheme, thus, allowing for the use of the efficient fast Fourier transform algorithm to recover the data. Then, we demonstrate how to use this new approach to control signal parameters in the case of the continuous spectrum.

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Introduction.—The nonlinear Schrödinger equation (NLSE) is a generic fundamental mathematical model with numerous applications in science and technology. In particular, the NLSE describes a path-average propagation of light in fiber-optic systems that is the backbone of the modern global communication networks. The NLSE is an example of a fundamental model of nonlinear physics, which can be integrated by the inverse scattering transform (IST) method [1,2]. The IST method is one of the greatest achievements of mathematical physics in the 20th century (see, e.g., [1–7] and references therein). In recent years (especially in optical communications), the IST method is also referred to as the nonlinear Fourier transform (NFT), stressing the similarity to the conventional Fourier transform and the ability of the IST/NFT to present solutions of the nonlinear evolution equation on the basis of noninteracting modes called scattering data or (in the NFT notation) *nonlinear spectrum*.

One specific, albeit highly important, application of the NLSE is in optical communications, where it is derived as a path-average (over periodic variations of power due to loss and gain) model governing the signal propagation along the transmission line [6,8,9] (here, we use the normalized version)

$$i\frac{\partial q}{\partial z} + \frac{1}{2}\frac{\partial^2 q}{\partial t^2} + |q|^2 q = 0. \tag{1}$$

Here, $q(z,t)$ is an optical field envelope, z is a propagation distance along the optical fiber, and t is the retarded time.

The general solution of the NLSE is presented by the superposition of solitary (localized in time) waves corresponding to the discrete (solitonic) part of the nonlinear spectrum and dispersive waves associated with the continuous part of the nonlinear spectrum. Recent advances in coherent optical communication allowing information coding both over the amplitude and phase have made it possible to reconsider relatively old ideas of using the soliton solution of the NLSE [6,9] and nonlinear spectrum eigenvalues for the transmission of information [10] in the new context. The recent surge of interest in nonlinear transmission techniques is, in particular, due to the observation that conventional (linear) data transmission techniques are facing serious challenges induced by the nonlinear properties of the optical fiber communication channels (an excellent overview is given in [11,12]). This calls for the development of new nonlinear techniques of signal coding, transmission, and processing.

The traditional soliton transmission has been recently reassessed in [13,14] in the context of coherent communication and the use of soliton phase for data transmission. Moreover, a great deal of interest has been sparked recently by the application of the powerful IST/NFT methods in optical communications (see, e.g., [15–21] and references therein; we simply are not able to review here all relevant papers that have been published recently in this fast-growing field).

The efficiency of numerical algorithms for data encoding or decoding is critically important in the digital telecommunication networks. For instance, in wireless communication, the success and popularity of the orthogonal frequency division multiplexing (OFDM) method is due to the exceptional computational performance and high spectral efficiency of the fast Fourier transform (FFT) [22].

The success of the practical implementation of the nonlinear IST/NFT techniques will be defined by the availability of the fast and superfast NFT methods [19,23,24] and the stability of algorithms with respect to noise impact. The IST/NFT technique is relatively new compared to conventional methods, and the currently available numerical algorithms of information encoding or decoding using a solitonic signal are still far from the efficiency required in practical hardware implementation.

Here, we propose to use the kernel of the Gelfand-Levitan-Marchenko equations (GLME) to encode information; in particular, we demonstrate that the OFDM scheme can be applied in an efficient way. To create a signal at the beginning of the transmission line, as well as to recover the encoded kernel at the end of the line, here, we use the efficient Toeplitz inner bordering (TIB) numerical scheme of inverse and direct scattering transform (which was recently introduced by Frumin and coauthors [23,24]) and the exact soliton solution known from the IST theory [1,2].

For the discrete nonlinear spectrum, we propose a soliton orthogonal frequency division multiplexing (SOFDM) technique that is based on the choice of identical imaginary parts of N-soliton solution eigenvalues, corresponding to equidistant soliton frequencies, making it similar to the conventional OFDM scheme and allowing the use of the efficient fast Fourier transform algorithm to recover the data. We also demonstrate how the concept of the OFDM can be applied for the continuous spectrum kernel [17]. The important advantage of using coding over kernel of the GLME is the possibility of controlling signal parameters by utilizing the time domain window functions in the modulated kernel.

N-soliton solutions of the NLSE for N-symbol block transmission.—In the traditional soliton transmission, a single (soliton) pulse is used as an information carrier sent over a time slot allocated to one symbol in a given spectral channel [6,9]. Transmission, in this case, is affected by the soliton interactions and/or is restricted in the spectral efficiency because a separate soliton occupies a small fraction of the symbol duration time. A great deal of attention has recently been placed on the potential use of the discrete nonlinear eigenvalues in fiber-optic channels. Most of the current studies of discrete nonlinear eigenvalue communications are limited to exploring different solitonic waveforms (forming a transmitted alphabet) in a single-symbol time slot. To avoid interaction between neighboring symbols, a long guard interval is typically used to suppress intersymbol interactions, thus, limiting spectral efficiency of such burst-mode transmission.

We propose here to use the well-known [1,2] general analytical N-soliton solutions of the NLSE (N-SS) (see the Supplemental Material [25]) for N-symbol block modulation and coding. In a block transmission technique, the information symbols are arranged in the blocks separated by some known symbols. Application of the N-SS allows one to simultaneously code information over N-symbol time intervals. Four soliton parameters, in principle, offer a possibility of four-dimensional modulation (coding) per soliton (symbol). Over the interval of N symbols, N-soliton solutions can offer $4 \times N$ degrees of freedom.

Recall that single soliton solutions read

$$q^{(1)}(z,t) = 2\beta \frac{\exp[-2i\omega t - 2i(\omega^2 - \beta^2)z + i\theta]}{\cosh(2\beta t + 4\omega\beta z - \delta t)}. \quad (2)$$

Here, obviously, 2β corresponds to soliton amplitude, 2ω is soliton frequency, θ is pulse phase, and $\delta t/(2\beta)$ defines soliton timing position. These four parameters can be used for the coding of information, i.e., amplitude, frequency, phase, and pulse position modulations, leading to various high-level modulation formats. Note that interactions between solitons are automatically accounted for in the N-soliton solution. Therefore, in N-SS coding, there is no issue of soliton interactions that occur when solitons are treated as separate entities.

The N-SS is defined by its scattering data or nonlinear spectrum: two sets of N complex constants. The first set corresponds to the complex eigenvalues of solitons $\xi_k = i\beta_k + \omega_k$, $k = 1, \ldots, N$. As discussed above, the imaginary part $\beta_k > 0$ defines corresponding (with the index k) soliton amplitude, and the real part ω_k is related to the soliton frequency (and corresponding group speed). The second set is given by the complex numbers $c_k = C_k \exp(i\theta_k)$, with real parameters C_k and θ_k. For the well-separated solitons, parameters C_k define timing positions of solitons in the following way: $\delta t_k = \ln[C_k/(2\beta_k)]$, while parameters θ_k define soliton phases. Based on the structure of the solitonic scattering data, the possible data coding of N-SS form two natural groups classified as *amplitude-frequency* modulation and *pulse position-phase* modulation. In general, there are $4 \times N$ free parameters that can be used for modulation.

The generation of a modulated (i.e., encoded) N-SS signal at the transmitter requires an algorithmic realization of IST/NFT in the encoder. Here, to find the N-SS, we use the standard factorization of the GLME, which leads to the well-known exact formulas (see the Supplemental Material [25]). Alternatively, the N-SS can be obtained by algebraic versions of IST, such as the Zakharov-Shabat dressing method [31], Darboux transformation [32], the method of Hirota [33], and by the IST TIB algorithm. Note that all these approaches are numerically unstable at large N, which limits their applications (see the Supplemental Material [25]). The kernel $\Omega(z,t)$ of the GLME for the N-SS has the following form:

$$\Omega^{(N)}(t,z) = \sum_{k=1}^{N} c_k(z) e^{-i\xi_k t}. \quad (3)$$

The sum in Eq. (3) is similar to the Fourier series; however, the "frequencies" ξ_k, in general, are complex numbers. Formally, the N-SS can be written as the IST/inverse NFT of the kernel (3)

$$q^{(N)}(z,t) = \mathrm{IST}[\Omega^{(N)}(z,t)]. \quad (4)$$

In what follows, for the sake of simplicity, the index (N) will be omitted. We assume that the coding or modulation is applied at

$z = 0$ (encoder) and decoding or demodulation (decoder) at $z = L$. The IST/NFT method links the nonlinear spectrum at $z = 0$ and $z = L$ by the following simple phase shift:

$$c_k(L) = c_k(0)\exp(-2i\xi_k^2 L), \qquad k = (1, \ldots, N). \quad (5)$$

Expressions (4) and (5) formally solve the problem of the compensation of signal distortions in the communication channels described by the NLSE. We believe, that formula (3) can offer some advantages for encoding or decoding operations, compared to the traditionally used formula (4).

Our central idea is to use the N-SS kernel (3) for the modulation of the information data. In this case, the number of numerical operations at the decoder is reduced. In the proposed scheme, the decoding operation requires one to recover only the kernel (3) at $z = 0$ by solving the direct scattering problem and by the application of a simple transformation (5). Moreover, as we will demonstrate, the analogy of the N-SS kernel (3) with the Fourier series allows us to introduce the OFDM scheme for the discrete spectrum.

Solitonic OFDM method.—In this section, we introduce a soliton OFDM (SOFDM) technique in which the GLME kernel can be efficiently used for encoding or decoding $2 \times N$ position-phase parameters. The key idea can be understood from the expression for the N-soliton kernel (3). The kernel would be similar to the conventional OFDM in case of real ξ_k. Therefore, we impose special conditions on the complex soliton parameters ξ_k. Namely, we consider N-SS, with eigenvalues $\xi_n = \omega_n + iA$. In this case, solitons have equal amplitudes but different equidistantly selected frequencies. Thus, the GLME kernel (3) at the beginning of the line is given by the finite Fourier series multiplied by e^{At}

$$\Omega(0, t) = e^{At}\sum_{k=1}^{N} c_k e^{-i\omega_k t}. \quad (6)$$

This greatly simplifies the processing of such signals.

Now, without loss of generality, we consider modulation over phase θ_n, while the pulse positions δ_n are left unmodulated. As a particular example of the SOFDM encoding, we consider $\tilde{N}$-phase-shift-keying ($\tilde{N}$-PSK) modulated N-SS. To apply the SOFDM over the finite time slot T, we introduce the discrete time grid

$$t_m = (m-1)T/N, \qquad m = 1, \ldots, N. \quad (7)$$

The orthogonality of Fourier harmonics is given by the following condition:

$$t_m\omega_n = 2\pi(m-1)(n-1)/N, \qquad m, n = 1, \ldots, N. \quad (8)$$

Similar to the standard OFDM, the FFT makes it possible to determine the parameters of signal modulation c_n by $O(N \ln N)$ arithmetic operations

$$c_n = \mathrm{FFT}[\Omega(0, t_m)\exp(-At_m)]. \quad (9)$$

To compute the scattering data from the received signal $q(t, L)$, one can use any available algorithm of the direct NFT. Here, without loss of generality, we use the direct TIB algorithm, calculating the entire signal kernel in the time domain by solving the GLME (see Supplemental Material [25] and [24]). The kernel contains all scattering data information: soliton eigenvalue positions (corresponding to amplitude and frequency modulations), pulse positions, and phases ($4 \times N$ parameters). Here, we focus only on a phase modulation to illustrate the proposed concept. The eigenvalue modulation is also possible, but it faces challenges in terms of efficiency and stability (see Supplemental Material [25]).

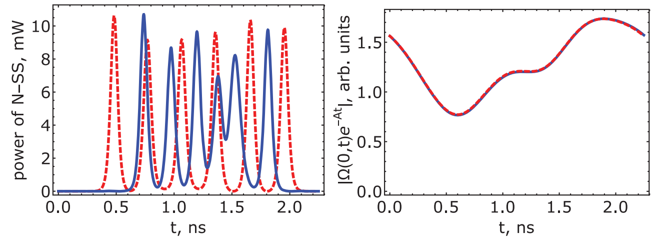

FIG. 1. Left: The modulus of the 6-soliton solution at the beginning (q_0, red dashed line) and at the end (q_1, blue solid line) of the NLSE governed transmission line. Right: 6-soliton normalized kernel of the GLME equations at the beginning of the transmission line $|\Omega(0, t)\exp(-At)|$. Blue solid line—the exact normalized kernel encoded by the SOFDM method with QPSK; red dashed line—the same kernel restored by the use of direct TIB method.

For illustration purposes, we choose the minimum possible number of time samples $M = N$. But actually, the value of M depends on the algorithm of the direct scattering transform at the receiver, and usually $M > N$. At the transmitter, we use the inverse fast Fourier transform (IFFT) to obtain the kernel (6) from the given data encoded by the phases c_n and then solve the inverse scattering problem as described in the Supplemental Material [25] to generate the input optical N-SS signal $q(0, t_m)$.

We test the SOFDM method in numerical simulations of data transmission by the use of quaternary phase-shift-keying (QPSK) modulated 6-soliton solution. In Fig. 1 (left), we present an example of a 6-soliton signal at the beginning and at the end of the transmission line of length $L = 2000$ km. Using the direct TIB method, we recover the encoded kernel that is presented in Fig. 1 (right). To avoid signal expansion, we arrange the solitons in order of descending velocity so that the slowest soliton occupies the first position in the signal, while the fastest soliton starts propagation from the signal end. However, we would like to stress that the practical implementation of the solitonic OFDM scheme requires further development of fast noise-stable methods for solving the direct scattering problem that we consider in the discussion section.

Kernel coding of the continuous nonlinear spectrum OFDM.—The kernel of the GLME for the continuous spectrum is presented in the following form:

$$\Omega^{(R)}(0, t) = R(0, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r_\omega e^{-i\omega t} d\omega. \quad (10)$$

Here, r_ω is the reflection coefficient of the Zakharov-Shabat system for the given signal $q^{(R)}(0, t)$, while $R(0, t)$ is commonly referred to as a signal response function. Similar

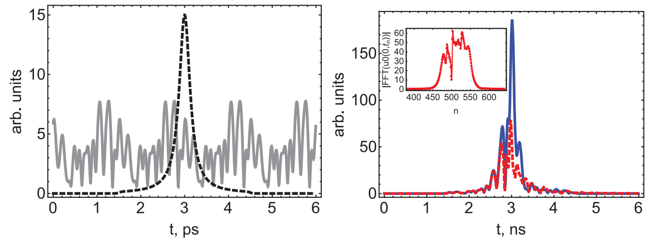

FIG. 2. Left: 16 encoded kernel harmonics (grey, solid line) and the window function (black, dashed line). Parameters of the window transformation function (12) are the following: $\tilde{A} = 15$, $\Gamma = 20$. Right: The double kernel (blue, solid line) obtained by multiplying the encoded harmonics by the window function and corresponding signal, obtained by the use of the inverse TIB algorithm (red, dashed line). The inset picture shows the absolute value of signal Fourier spectrum; the frequency index n is defined in (8).

to the discrete spectrum case, the IST links the continuous spectrum at $z = 0$ and $z = L$ by the following relation:

$$r_\omega(L) = r_\omega(0)\exp(-2i\omega^2 L). \tag{11}$$

The general idea to apply the OFDM scheme to the continuous nonlinear spectrum was previously considered in the framework of the so-called "nonlinear inverse synthesis" method [17,34]. This approach, while promising, has an important challenge—how to control the signal characteristics in the time domain. Indeed, the reflection coefficient $r(\omega)$, which was chosen for encoding information, is nontrivially coupled with the signal via IST.

Here, we propose to apply the additional window transformation to the kernel $\Omega^{(R)}(0,t)$ in the time domain as a method of controlling signal parameters. For IST-based schemes, the strong localization of the signal in time slots is highly critical to avoid nonlinear interactions between neighboring symbol intervals. Bearing in mind the linear limit $[q^{(R)}(z,t) \to 2\Omega^{(R)}(z,t)]$, we conclude, that well-localized (in time domain) signals should correspond to the localized kernel, at least in a weakly nonlinear case. We have examined different window transformation functions known from the linear communication theory (see, for instance, [35]) and found that the excellent signal localization in time is achieved for window functions with smooth polynomial fronts.

Figure 2 demonstrates signal generation at the beginning of the transmission line. We start from 16 Fourier harmonics encoded using the OFDM 8-PSK scheme. Then, we apply the window transformation $f(t)$, similar to the well-known Lorentzian function

$$F(t) = \frac{\tilde{A}}{[\Gamma(t-t_0)]^2 + 1} \tag{12}$$

to localize the signal in the time slot. Here $\tilde{A}$, Γ are the parameters of the window transformation corresponding to the characteristic amplitude and width of the modulated kernel, and t_0 corresponds to the center of the time slot. We also add to the window transformation function (12) exponentially decaying tails, which do not affect the general shape of the signal but help to cancel interactions between neighboring bursts. Finally, we find the signal profile using the inverse TIB method (Fig. 2, right, red).

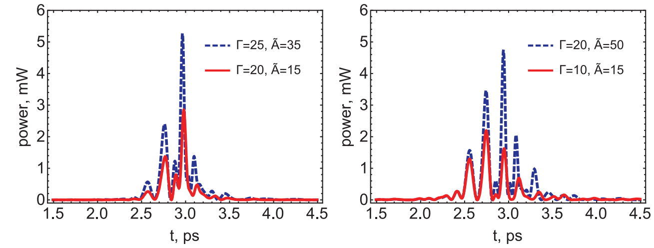

FIG. 3. Dependence of the signal from the parameters of the kernel window transformation function (12).

Next, we study the dependence of signal shape on the parameters $\tilde{A}$ and Γ of the modulated kernel. We have found that varying the kernel window function parameters allows us to control the characteristics of the generated signal without affecting the information content, as illustrated in Fig. 3. Figure 4 presents the results of numerical modeling of a burst-mode signal transmission in the NLSE channel (with noise), with a total propagation distance L of 1000 km, and SNR = 19.7 dB. The modeling was performed using the standard split-step method with adding noise at each numerical spatial step that corresponds to the distributed noise model (see, e.g., [11,12]). We have found that the direct TIB algorithm remains stable for considered SNR values. Interestingly, better decoding results can be achieved using only the first (left) part of the pulse. This is not surprising since the direct TIB algorithm successively recovers the kernel from the left to the right end of the signal. Thus, the right part of the recovered kernel is additionally affected by noise distortions, accumulated during the calculation of the left kernel part.

Discussion and conclusion.—In this Letter, we proposed and examined new approaches to coding information over

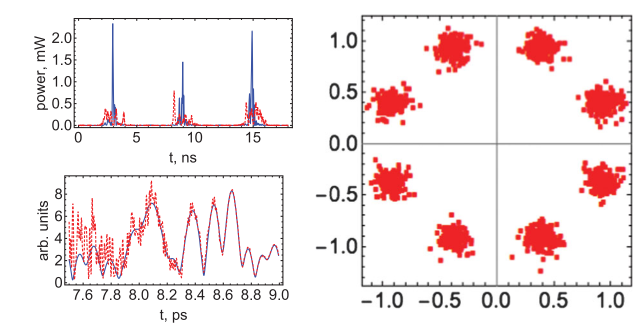

FIG. 4. Left: Propagation of the burst-mode signal in the model NLSE channel, with $L = 1000$ km and SNR = 19.7 dB. Blue solid lines correspond to the signal at the beginning of the line; red dashed lines show the signal at the end of the line. Parameters of the window transformation function are the same as in Fig. 2. The bottom picture corresponds to the encoded (blue, solid lines) and decoded, using the direct TIB algorithm (red, dashed lines) kernel harmonics for the central burst interval. Right: Constellation diagram for the central burst interval (statistics on a 10^3 randomly encoded initial signals).

the kernel of the GLME. We have considered both the discrete (solitonic) and continuous part of scattering data. We demonstrated that application of the direct TIB method allows one to recover the most stable part of the kernel, which is an advantage in the presence of distributed noise.

We have proposed, to the best of our knowledge for the first time, to use the general N-soliton solution of the NLSE for simultaneous coding of N symbols involving $4 \times N$ coding parameters, instead of separate N solitons. As a particular subclass of the general schemes, we examined a soliton orthogonal frequency division multiplexing technique that is based on the choice of identical imaginary parts of N-soliton solution eigenvalues, corresponding to equidistant soliton frequencies, making it similar to the conventional OFDM scheme. This allows us to use the efficient fast Fourier transform algorithm to recover the data. We would like to point out that efficient implementation of numerical recovery of solitonic kernels by solving GLME requires the development of numerical algorithms, which are stable against additive noise.

For the continuous spectrum, we have tested the stability of the direct TIB method against the additive noise and proposed to use the localized kernel in the time domain to control properties of the corresponding generated signal. The latter can be considered as a novel realization of the "nonlinear inverse synthesis" method [17,34].

We demonstrated that the mathematical properties of the NLSE can be used for introducing fundamentally novel (compared to the linear communication theory) methods for the coding and detection of the signal setting foundation for the nonlinear communication theory.

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[1] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].

[2] M. J. Ablowitz, D. J. Kaup, and A. C. Newell, Stud. Appl. Math. **53**, 249 (1974).

[3] S. Novikov, *Theory of Solitons: The Inverse Scattering Method* (Springer Science & Business Media, New York, 1984).

[4] A. C. Newell, *Solitons in Mathematics and Physics* (Society for Industrial and Applied Mathematics, Philadelphia, 1985).

[5] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (Society for Industrial and Applied Mathematics, Philadelphia, 1981), Vol. 4.

[6] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications* (Oxford University Press, New York, 1995).

[7] G. L. Lamb, *Elements of Soliton Theory* (Wiley-Interscience, New York, 1980).

[8] Y. S. Kivshar and G. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, San Diego, 2003).

[9] L. F. Mollenauer and J. P. Gordon, *Solitons in Optical Fibers: Fundamentals and Applications* (Academic Press, Boston, 2006).

[10] A. Hasegawa and T. Nyu, J. Lightwave Technol. **11**, 395 (1993).

[11] R.-J. Essiambre, G. J. Foschini, G. Kramer, and P. J. Winzer, Phys. Rev. Lett. **101**, 163901 (2008).

[12] R.-J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, J. Lightwave Technol. **28**, 662 (2010).

[13] J. E. Prilepsky, S. A. Derevyanko, and S. K. Turitsyn, Phys. Rev. Lett. **108**, 183902 (2012).

[14] O. Yushko, A. Redyuk, M. Fedoruk, and S. Turitsyn, J. Exp. Theor. Phys. **119**, 787 (2014).

[15] E. G. Turitsyna and S. K. Turitsyn, Opt. Lett. **38**, 4186 (2013).

[16] M. I. Yousefi and F. R. Kschischang, IEEE Trans. Inf. Theory **60**, 4312 (2014).

[17] J. E. Prilepsky, S. A. Derevyanko, K. J. Blow, I. Gabitov, and S. K. Turitsyn, Phys. Rev. Lett. **113**, 013901 (2014).

[18] S. T. Le, I. D. Philips, J. E. Prilepsky, P. Harper, A. D. Ellis, and S. K. Turitsyn, J. Lightwave Technol. **34**, 2459 (2016).

[19] S. Wahls and H. V. Poor, IEEE Trans. Inf. Theory **61**, 6957 (2015).

[20] S. A. Derevyanko, J. E. Prilepsky, and S. K. Turitsyn, Nat. Commun. **7**, 12710 (2016).

[21] V. Aref, S. T. Le, and H. Buelow, in *Proceedings of 42nd European Conference on Optical Communication* (VDE, Duesseldorf, 2016), pp. 1–3.

[22] W. Shieh and I. Djordjevic, *OFDM for Optical Communications* (Academic Press, Amsterdam, 2009).

[23] O. Belai, L. Frumin, E. Podivilov, and D. Shapiro, J. Opt. Soc. Am. B **24**, 1451 (2007).

[24] L. L. Frumin, O. V. Belai, E. V. Podivilov, and D. A. Shapiro, J. Opt. Soc. Am. B **32**, 290 (2015).

[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.118.223901, which includes Refs. [26–30] for the basic information concerning numerical approaches for inverse and direct scattering transform and parametric kernel decoding.

[26] A. Gelash and D. Agafontsev, in *Proceedings of EGU General Assembly, Vienna, Austria, 2017* (2017), Vol. 19, p. 15446.

[27] E. E. Tyrtyshnikov, Numer. Math. **67**, 261 (1994).

[28] R. E. Blahut, *Fast Algorithms for Signal Processing* (Cambridge University Press, Cambridge, 2010).

[29] I. Gohberg and V. Olshevsky, J. Complexity **13**, 208 (1997).

[30] S. L. Marple, *Digital Spectral Analysis with Applications* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1987).

[31] V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. **8**, 226 (1975); **13**, 166 (1979).

[32] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer-Verlag, Berlin, 1991).

[33] R. Hirota, J. Math. Phys. (N.Y.) **14**, 810 (1973).

[34] S. T. Le, J. E. Prilepsky, and S. K. Turitsyn, Opt. Express **23**, 8317 (2015).

[35] G. M. Jenkins and D. G. Watts, *Spectral Analysis* (Holden-Day, San Francisco, 1968).