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On the theory of adiabatic field dynamics in the Kerr medium with distributed gain and dispersion

SERGEI K. TURITSYN[1,2]

[1]Aston Institute of Photonic Technologies, Aston University, Birmingham, B4 7ET, UK
[2]Novosibirsk State University, Novosibirsk 630090, Russia (s.k.turitsyn@aston.ac.uk)

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A general theory is presented for the adiabatic field evolution in a nonlinear Kerr medium with distributed amplification and varying dispersion. Analytical expression is derived linking parameters of the adiabaticity, gain distribution, and dispersion profile. As a particular example, an optical pulse compressor based on the adiabatic dynamics is examined.

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Nonlinear optical systems offer a number of practical applications based on nonlinear science concepts, ranging from solitons and supercontinuum to nonlinear Fourier transform and parametric amplification. Interplay between dispersion and nonlinearity, combined with distributed amplification, creates a number of opportunities for experimental implementation of fascinating nonlinear dynamics. Fiber-optic is especially attractive due to an excellent waveguiding and relative simplification of the corresponding underlying models [1,2]. This Letter deals with the general mathematical theory of the adiabatic optical pulse evolution in the nonlinear Kerr medium with gain and dispersion, but without loss of generality. The fiber-optic notations are used and a specific pulse compression example is examined in what follows.

There are several key methods of laser pulse compression using fiber optics. First is the so-called fiber grating compression approach that exploits normal dispersion fiber dynamics [3,4]. Propagation in the normal dispersion spectrally broadens pulses and produces via the self-phase modulation effect a quasi-linear (which is exactly linear for parabolic pulses) temporal chirp. At the next stage, the pulses are compressed up to the temporal widths determined by its spectral bandwidth using elements with anomalous dispersion.

The second approach is the so-called multi-soliton compression technique [5], which is based on higher-order nonlinear soliton dynamics in anomalous dispersion. Evolution of the higher-order solitons along the fiber is characterized by periodic compression of the input state leading to a narrow temporal spike. This nonlinear compression method allows for strong compression, but requires a careful adjustment of the input power to control the point of maximal compression. Another well-known challenge in this technique is the appearance of a broad pedestal that contains a large fraction of the total energy. The advantage of active fiber-based compressors is the possibility to use low-power pulse sources and a combination of compression and amplification [6–8]. Amplifying nonlinear effect compressors can enhance both fiber grating compression and multi-soliton methods. For instance, in the amplifying fiber with normal dispersion, the input pulse evolves into a cleaner linearly chirped parabolic pulse, which can be efficiently compressed [7].

The possibility to use varying dispersion [9] in fiber provides additional opportunities for pulse compression. One of the attractive and robust compression schemes is adiabatic soliton compression in the dispersion-decreasing fiber, which ensures time-bandwidth-limited output [10–16]. Tapering can be applied to gas-filled hollow-core anti-resonant fibers to achieve generation of extreme ultraviolet dispersive waves [17]. Strict mathematical conditions for adiabatic soliton dynamics have been presented for an axially nonuniform (tapered) fiber in [10]. Most fiber-based dispersion-tailored compressors use tapered fiber spans with varying diameter that changes simultaneously dispersion, effective nonlinear coefficient, and effective refraction index. The adiabatic compression technique is based on the perturbation theory of the nonlinear Schrödinger equation (NLSE), which ensures smooth soliton evolution that preserves some integral pulse characteristics. Mathematically, the adiabatic condition is that the product of the effective gain (or loss) coefficient and the soliton period is less than unity. This condition will be discussed in detail below. True adiabatic compression does not produce any extra chirp, maintaining transform-limited pulses.

In this Letter, mathematical theory of the adiabatic pulse compression in a nonlinear Kerr medium with both varying dispersion and distributed amplification are examined. Though our analysis has some similarity to the tapered fiber [10,11], we consider a different and more generic design that can be applied beyond the tapered waveguide systems. Note that this general consideration does not include effects such as, e.g., higher-order dispersion, Raman scattering, and other effects that limit applications of the approach. Typically, these limiting effects vary from one application to another and are less general than the considered generic system. Therefore, we leave analysis of the limiting impact of the higher-order terms beyond the scope of this Letter. It is interesting to point out that in some situations,

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the detrimental effect of Raman scattering and higher-order dispersion can interfere, compensating each other and keeping the pulse compression adiabatic [18].

Optical field $E(z,t)$ propagation down the amplifying optical medium with Kerr nonlinearity and varying group velocity dispersion is governed by the generalized NLSE with loss and gain (for details, see, e.g., [1,2,19]):

$$i\frac{\partial E}{\partial z} - \frac{\beta_2(z)}{2}\frac{\partial^2 E}{\partial t^2} + \gamma|E|^2 E = -i\alpha E + ig(z)E. \quad (1)$$

Here, $\beta_2(z) = -|\beta_2(0)|s^2(z)$ is the group velocity dispersion, and dimensionless function $s^2(z)$ (normalized with the condition $s(0) = 1$) defines variations of dispersion profile along the waveguide; we consider here only anomalous dispersion (in fiber-optic terms) with $\beta_2 < 0$. γ is the nonlinear Kerr coefficient, α is linear loss, and $g(z)$ describes a distributed gain. Consider waveforms (e.g., pulses) with the characteristic temporal duration T_0. It is assumed here that the gain distribution along the waveguide is given, and our goal is to determine the longitudinal dispersion profile that provides an effective adiabatic propagation of optical pulse with a given power level.

Let us transform Eq. (1) using the following substitution and a straightforward change of variables:

$$E(z,t) = \sqrt{\frac{|\beta_2(0)|}{\gamma T_0^2}} \times s(z) \times q(Z,T), \quad (2)$$

where

$$Z = \frac{\int_0^z s^2(z')\mathrm{d}z'}{L_{\mathrm{dis}}}, \qquad L_{\mathrm{dis}} = \frac{T_0^2}{|\beta_2(0)|}, \qquad T = \frac{t}{T_0}. \quad (3)$$

Consider a general design of the dispersion profile varying with z that can provide for the adiabatic evolution with the targeted adiabaticity parameter $\epsilon \ll 1$ for any given gain distribution $g(z)$. It is easy to check that by selecting a normalized dispersion $s(z)$ that satisfies the equation

$$\frac{ds}{dz} = -\frac{\epsilon}{L_{\mathrm{dis}}} s^3 + [g(z) - \alpha]s, \quad (4)$$

we can ensure that the dynamics of the field $q(Z,T)$ is governed by the following master equation that presents classical NLSE with linear perturbation:

$$i\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^2 q = i\epsilon q. \quad (5)$$

Assuming ϵ to be small, this equation describes adiabatic evolution of the field $q(Z,T)$ with Z. This is a master model for waveguide systems based on the adiabatic signal dynamics. One of the advantages of the adiabatic approach is the possibility to use a powerful mathematical tool for system design. The NLSE [i.e., Eq. (5) with $\epsilon = 0$] is integrable by the so-called inverse scattering transform (IST) method [20] (also known as the nonlinear Fourier transform; see, e.g., a recent review of its applications in optical communications [21]).

In the case of small $|\epsilon| \ll 1$, optical field evolution can be analyzed using the perturbation theory based on the IST [19,22]. Recall a well-known fact that for a single soliton having initial shape $q(T, Z=0) = \lambda_0/\cosh[\lambda_0 T]$, the adiabatic evolution is given by (see, e.g., [2,19])

$$q(T,Z) = \frac{\lambda(Z) \times \exp[i\sigma(Z)]}{\cosh[\lambda(Z)T]}, \quad (6)$$

where evolution of a soliton amplitude is given by $\lambda(z) = \lambda_0 \exp[2\epsilon Z]$ and phase by $\sigma(Z) = \lambda_0^2 \times (1 - \exp[4\epsilon Z])/(8\epsilon)$. In dimension units, an input pulse power is determined as $P_{in} = \lambda_0^2|\beta_2(0)|/(\gamma T_0^2)$.

Evolution of a pulse full width at half-maximum T_{FWHM} and corresponding broadening or compression is given by $T_{\mathrm{FWHM}}(Z) = 1.763 \exp[-2\epsilon Z]/\lambda_0$. It is seen that positive ϵ (effective gain) leads to temporal pulse compression, while negative ϵ (effective loss) corresponds to temporal broadening and spectral compression. This is the basics of the adiabatic compression of soliton pulse in the media described by Eq. (5). One can see that parameter ϵ defines the rate of adiabatic pulse compression. It can be controlled by appropriate design of the dispersion profile for a given distributed gain $g(z)$. It is important to point out that adiabatic evolution is not linked to single-soliton dynamics and might be exploited for various input waveforms.

A general solution of Eq. (4) for the normalized dispersion profile $s^2(z)$ with the initial condition $s^2(0) = 1$ is found as

$$s^2(z) = \frac{\exp[F(z)]}{1 + \frac{2\epsilon}{L_{\mathrm{dis}}}\int_0^z \exp[F(z')\mathrm{d}z']}, \quad (7)$$

where

$$F(z) = 2\int_0^z (g(z') - \alpha)\mathrm{d}z'. \quad (8)$$

Equations (7) and (8) present the main result of the Letter. They give a profile of the longitudinal dispersion variation $s(z)$ that for a given gain distribution allows to achieve a strictly adiabatic regime governed by Eq. (5). This expression generalizes the result for the dispersion profile tailored to the gain to achieve lossless NLSE propagation [23]. In the case of the lossless NLSE ($F(z) = 0$), Eq. (7) reproduces a well-known result: $s^2(z) = 1/(1 + 2\epsilon\, z/L_{\mathrm{dis}})$. Note that though ϵ is small, the second term in the denominator is not necessarily also small. Using Eqs. (7) and (8), we can explicitly express the variable Z as

$$Z = \frac{\int_0^z s^2(z')\mathrm{d}z'}{L_{\mathrm{dis}}} = \frac{1}{\epsilon}\ln\left[1 + \frac{2\epsilon}{L_{\mathrm{dis}}}\int_0^z \exp[F(z')\mathrm{d}z']\right]. \quad (9)$$

Note that distributed gain can be implemented in various ways. Below, several particular examples of the application of a general theory are examined.

Now consider several specific examples of the compressor design. Relatively short spans of an active waveguide can provide with good accuracy a constant (uniform along the waveguide length) gain g_0. In the case of a constant loss or gain (e.g., in active waveguide) $g(z) - \alpha = g_0$, the expression for dispersion profile $s(z)$ is simplified:

$$s^2(z) = \frac{\exp[2g_0 z]}{1 + \epsilon(\exp[2g_0 z] - 1)/(g_0 L_{\mathrm{dis}})}. \quad (10)$$

For $\epsilon = 0$, from here we recover the well-known result by Tajima [9]: $s^2 = \exp[2g_0 z]$. In the dimension units, the optical field reads

$$E(z,t) = \sqrt{\frac{|\beta_2(0)|}{\gamma T_0^2}} \times \frac{\exp[g_0 z] \times q\left(Z, \frac{t}{T_0}\right)}{(1 + \epsilon[\exp(2g_0 z) - 1]/(g_0 L_{\mathrm{dis}}))^{1/2}}, \quad (11)$$

where $\epsilon Z = \ln[1 + \epsilon(\exp[2g_0 z] - 1)/(g_0 L_{\mathrm{dis}})]$. For a soliton, pulse power evolution down the waveguide is given by

$$|E(z,t)|^2 = \frac{P_s(z)}{\cosh^2\left[\frac{t}{\tau(z)}\right]}, \quad (12)$$

with varying pulse width parameter $\tau(z)$

$$\tau(z) = \frac{T_0}{\lambda_0[1+\epsilon[\exp(2g_0 z)-1]/(g_0 L_{\mathrm{dis}})]^2} \quad (13)$$

and power

$$P_s(z) = \frac{\lambda_0^2|\beta_2(0)|}{\gamma T_0^2} \times \exp(2g_0 z)\left(1+\frac{\epsilon[\exp(2g_0 z)-1]}{g_0 L_{\mathrm{dis}}}\right)^3. \quad (14)$$

It is seen from these relations that the considered adiabatic regime is more efficient with lower L_{dis} (e.g., higher anomalous dispersion). Note the exponential growth factor in Eq. (10) that dominates the first part of the $s^2(z)$ change with z. The term with ϵ that defines adiabatic evolution contributes at the point where gain makes the term $\epsilon[\exp(2g_0 z)-1]/(g_0 L_{\mathrm{dis}})$ comparable with unity. It is seen that the adiabatic compression requires either high variation of dispersion (in the case of not small L_{dis}) or small L_{dis} corresponding to short pulse and/or high level of dispersion. Once more, we would like to stress that we do not aim here to present any particular implementation, but rather a mathematical theory for a family of devices that can use different platforms. Figure 1 depicts a normalized dispersion profile $s^2(z)$ for $\epsilon = 0.2$, $g_0 = 0.23\ \mathrm{m}^{-1}$ (2 dB/m), and $L_{\mathrm{dis}} = 100$ m. The level of compression with propagation is also shown for the pulse width at half maximum $T_{\mathrm{FWHM}}(z)/T_{\mathrm{FWHM}}(0) = 1/(1+\epsilon[\exp(2g_0 z)-1]/(g_0 L_{\mathrm{dis}}))^2$. It is seen that smaller gain allows to keep dispersion variation within smaller range. Figure 2 presents similar plots, but for $g_0 = 0.115\ \mathrm{m}^{-1}$ (1 dB/m) and $L_{\mathrm{dis}} = 30$ m with all other parameters the same.

In the case of the often used backward Raman amplification, for unsaturated gain, a function $g(z)$ is given by $g(z) = 0.5 g_R P_{pb} \exp[-2\alpha_p(L-z)]$, where g_R is the Raman gain coefficient, P_{pb} is the backward pump power injected at $z = L$, and α_p is the loss at the pump wavelength. In this case,

$$F_b(z) = \frac{g_R P_{pb}\ \exp(-2\alpha_p L)}{2\alpha_p} \times [\exp(2\alpha_p z)-1] - 2\alpha z. \quad (15)$$

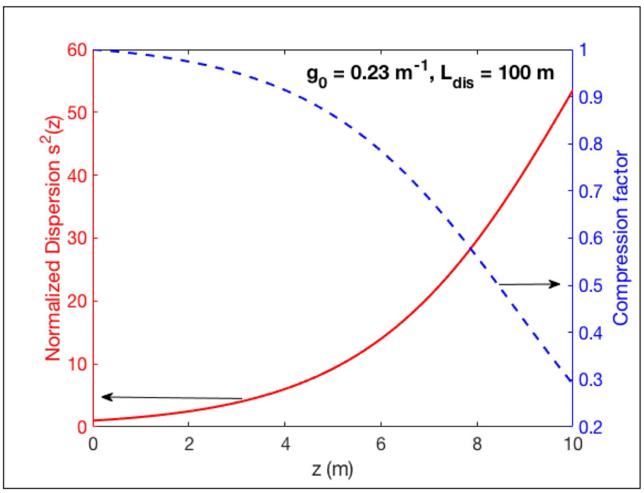

Fig. 1. Normalized dispersion $s^2(z)$ for $\epsilon = 0.2$, $g_0 = 0.23\ \mathrm{m}^{-1}$ (2 dB/m), $L_{\mathrm{dis}} = 100$ m, and compression factor of the pulse width at half maximum $T_{\mathrm{FWHM}}(z)/T_{\mathrm{FWHM}}(0) = 1/(1+\epsilon[\exp(2g_0 z)-1]/(g_0 L_{\mathrm{dis}}))^2$ with distance.

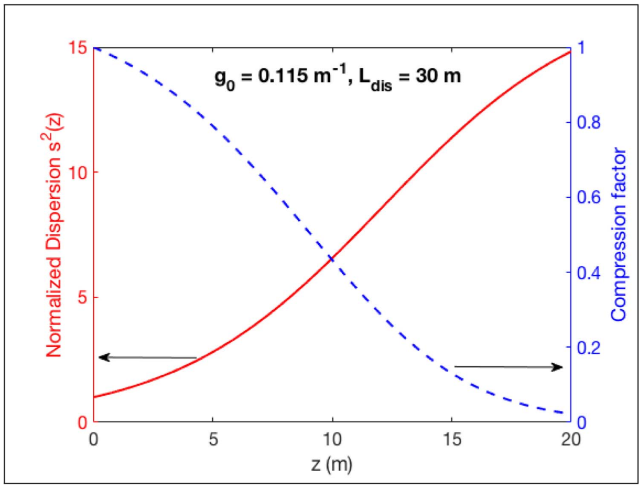

Fig. 2. Normalized dispersion $s^2(z)$ for $\epsilon = 0.2$, $g_0 = 0.115\ \mathrm{m}^{-1}$ (1 dB/m), $L_{\mathrm{dis}} = 30$ m, and compression factor of the pulse width at half maximum $T_{\mathrm{FWHM}}(z)/T_{\mathrm{FWHM}}(0) = 1/(1+\epsilon[\exp(2g_0 z)-1]/(g_0 L_{\mathrm{dis}}))^2$ with distance.

We would like to stress that in the considered scheme, Raman gain itself is not required to be adiabatic (small), but it is a combination of the distributed gain and varying dispersion that leads to the adiabatic dynamics.

Figure 3 shows a normalized dispersion profile $s^2(z)$ computed for the following parameters for the SMF-28: $\alpha = 0.023\ \mathrm{km}^{-1}$ at 1550 nm, $g_R = 0.4\ (\mathrm{W}\times\mathrm{km})^{-1}$, $\alpha_p = 0.0285\ \mathrm{km}^{-1}$, several values of $P_{pb} = 1$, 3, 5 W, $\epsilon = 0.2$ and $L_{\mathrm{dis}} = 5$ km.

In the similar manner we can derive expressions for the cases of forward and combined forward and backward distributed Raman amplification schemes. Unsaturated forward pumped Raman amplification yields

$$F_f(z) = \frac{g_R P_{pf}}{2\alpha_p} \times [1-\exp(-2\alpha_p z)] - 2\alpha z. \quad (16)$$

A combined forward and backward pumping scheme gives

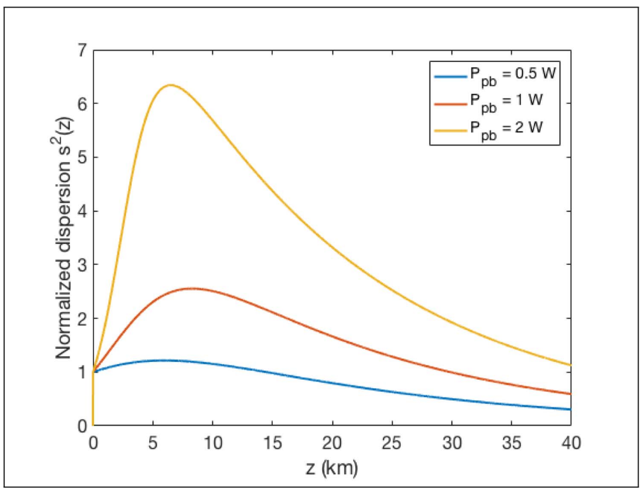

Fig. 3. Normalized dispersion profile $s^2(z)$; here, $\alpha = 0.023\ \mathrm{km}^{-1}$ at 1550 nm, $g_R = 0.4\ (\mathrm{W}\times\mathrm{km})^{-1}$, $\alpha_p = 0.0285\ \mathrm{km}^{-1}$; backward pump powers are $P_{pb} = 0.5$, 1, 2 W, $\epsilon = 0.2$ and $L_{\mathrm{dis}} = 5$ km.

$$F_{bf}(z) = \frac{g_R P_{pf}\left(1 - e^{-2\alpha_p z}\right) + g_R P_{pb}\left(e^{2\alpha_p (z-L)} - e^{-2\alpha_p L}\right)}{2\alpha_p} - 2\alpha z. \quad (17)$$

We would like to stress that distributed Raman amplification alone can be exploited for adiabatic compression. However, in this case, the condition of adiabaticity imposes severe limitations of the level of amplification. Adjusting the dispersion variation profile to the chosen amplification scheme allows one to relax these restrictions.

Let us recall, for completeness and comparison, a theory of adiabatic pulse soliton compression in the dispersion-decreasing tapered fiber. The equation governing propagation of the envelope $U(z,t)$ of light wave in nonuniform fiber reads (see details of notations and definitions in [10,11])

$$i\frac{\partial U}{\partial z} - \frac{k_2(z)}{2}\frac{\partial^2 U}{\partial t^2} + \gamma(z)|U|^2 U = i(\tilde{g}(z) - \alpha)U - \frac{i}{2f}\frac{df}{dz}U. \quad (18)$$

Here, $k_2(z)$ is a varying group velocity dispersion, $f(z)$ corresponds to variation of the effective refractive index, and $\gamma(z)$ describes gradual change of the effective mode area along the fiber [10,11,15]. The term with $f(z)$ ensures the conservation of the total photon number in nonuniform optical fiber in the absence of signal attenuation. For the sake of clarity, we do not include here the mean frequency variation effect, which can be added in a straightforward way following detailed description in [11]. Applying similar transform,

$$U(z,t) = \sqrt{\frac{|k_2(0)|}{\gamma(z)T_0^2}} \times s(z) \times w(\eta, T), \quad (19)$$

where

$$\eta = \frac{\int_0^z s^2(z')\mathrm{d}z'}{L_{\mathrm{dis}}}, \qquad L_{\mathrm{dis}} = \frac{T_0^2}{|k_2(0)|}, \qquad T = \frac{t}{T_0}, \quad (20)$$

one can derive adiabatically perturbed NLSE:

$$i\frac{\partial w}{\partial \eta} + \frac{1}{2}\frac{\partial^2 w}{\partial T^2} + |w|^2 w = i\epsilon w. \quad (21)$$

In this case, the expression for the dispersion profile $s(z)$ reads

$$\frac{ds}{dz} = -\frac{\epsilon}{L_{\mathrm{dis}}} s^3 + \left[g(z) - \alpha - \frac{1}{2f}\frac{df}{dz} + \frac{1}{2\gamma}\frac{d\gamma}{dz}\right] s. \quad (22)$$

The analytical solution given by Eqs. (7) and (8) can also be used in this case. Distributed amplification can be exploited to provide additional degrees of freedom for design of such nonuniform fiber-based adiabatic compression systems.

In conclusion, a mathematical theory is presented that links optical systems with Kerr nonlinearity, distributed amplification, and varying dispersion to an integrable model with strictly adiabatic field evolution. Though the focus of this Letter is on optical pulse compression, the theory is not limited to soliton dynamics, compression, or monotonic change in dispersion. It can also be used for applications with periodic changes of dispersion and gain. Analytical expression is derived for the dispersion profile that links parameters of adiabaticity (related to the compression rate), gain distribution, and input pulse parameters. It should be stressed that the considered theory is limited by various higher-order effects that should be taken into account when pulse becomes narrow. Consideration of the higher-order effects is straightforward [11,18]; however, it is beyond the scope of this Letter.

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